



**1. Title of Case Study:**  
**Regularity of classical three-dimensional fluids**

**2. Grant Reference Number or Facility Name:**  
**MidPlus**

**3. One sentence summary:**  
Massively parallel simulations of the interaction and reconnection of vortex structures have provided new insight into one of the outstanding mathematical questions

**4. One paragraph Summary**

This question is not simply of academic interest as the Navier-Stokes equations are believed to underlie most fluid motion. This includes the turbulent motion found in flows ranging from combustion in engines, air flow around bodies and on to the motion of the atmosphere and ocean. The difficulties arise from how the incompressible, rotational and turbulent nature that is either in the flows or around them. For example, if you squeeze it one way, it will slip out the other. There are also rotating eddies that can, collide and annihilate one another, and in the process dissipate the turbulent flow that created them. However, a universal theory that would apply all turbulent flows does not exist. Knowing whether there, or are not, singularities and the nature of the most intense events at the smallest scales could be the key to identifying what such a universal theory would look like. The Mid-Plus calculations leading to my two sole-authored 2013 papers showed two seemingly incompatible trends: For initial configuration that is the most liable to singular behaviour, the inviscid Euler equations have very strong but non-singular exponential of exponential growth at late stages. In contrast, Navier-Stokes calculations with the same initial conditions but different viscosities, developed turbulent spectra in a finite time, indicating some type of underlying singularity. New Navier-Stokes calculations that have extended the anti-parallel calculations to viscosities that are small enough to be determine not only what additional factors can limit any singular growth, but also how these effects might be mitigated. These calculations also showed the limitations in using the anti-parallel configuration to address these questions. The latest trefoil vortex knot calculations are one way around those limitations and can, in a controlled manner, examine how vortices link and unlink. The preliminary calculations show that it is this linking, or helicity, that is controlling the small-scale dynamics. Not the global energy, mean-square vorticity or local maxima in those fields. And going back to the anti-parallel results, can explain some of their features as well.

**5. Key outputs in bullet points:**

- Kerr R.M. 2013: Swirling, turbulent vortex rings formed from a chain reaction of reconnection events, Phys. Fluids 25, 065101
- Kerr, R.M. 2013: Bounds for Euler from vorticity moments and line divergence J. Fluid Mech. 729, R2.
- Donzis, D., Gibbon, J.D, Gupta, A., Kerr, R.M., Pandit, R., Vincenzi, D., 2013: Vorticity moments in four numerical simulations of the 3D Navier-Stokes equations. J. Fluid Mech. 732, 316-331.
- Kerr, R.M. 2015: Simulated Navier-Stokes trefoil reconnection. arXiv:1410.1259v1.

## 6. Main body text

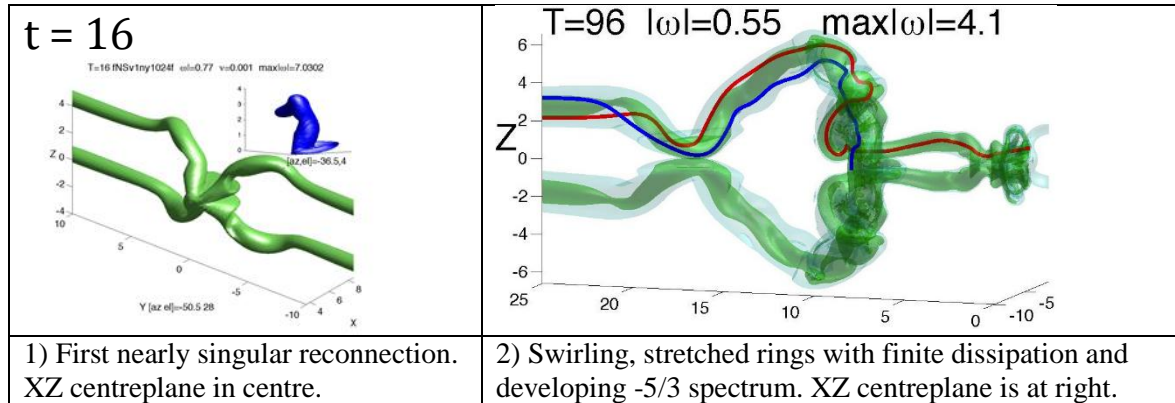
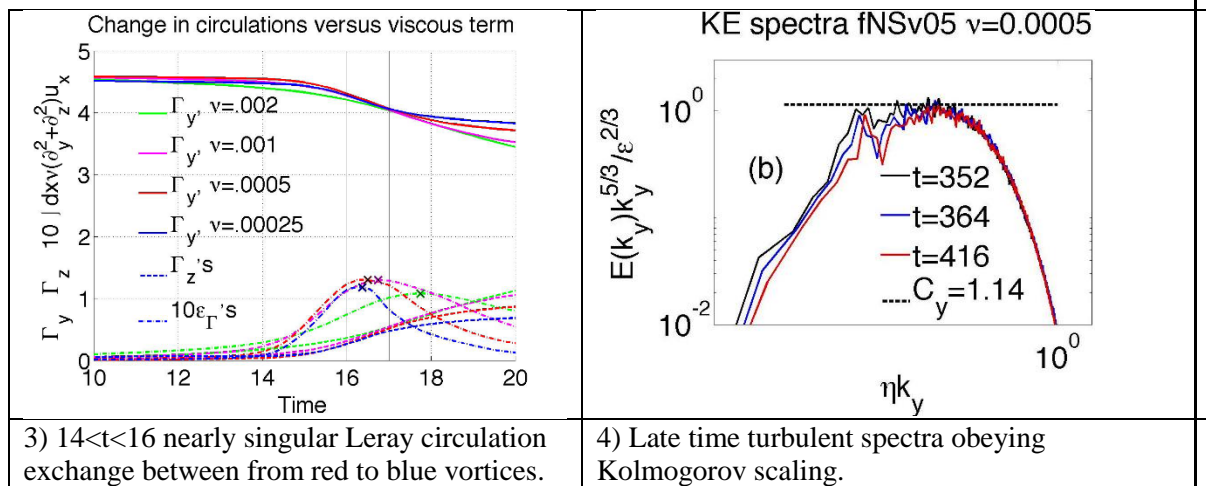


Figure 1 shows two initially anti-parallel vortices touching where the XY dividing plane and XZ symmetry plane meet. This is for both inviscid Euler from Kerr, JFM (2013) and Navier-Stokes in Kerr, Phys. Fluids (2013). Unlike previous anti-parallel reconnection calculations, this interaction zone has been isolated far from the periodic boundaries, which otherwise suppress singular trends. It also allows a chain (in y) of large rings, in y-z, to form. Stretching due to the multiple reconnections sustains a cascade to small scales, eventually resulting in the  $k^{-5/3}$  spectrum in Fig. 4 to form. Having to simulate large domains, plus the mesh refinement needed to resolve reconnection close to the XY dividing plane, requires very large computational domains of the type allowed by Mid-Plus



An important feature of all of the new simulations is the stability of the initial vorticity profiles against anything except large-scale perturbations. This allows detailed comparisons of small-scale statistics between cases with different meshes, to determine the resolution limits, as well as different Reynolds numbers, to determine the dependence of scaling properties upon the viscosity. Figure 3 shows the finest statistic yet, the  $y = z = 0$   $x$ -line integrals of the second velocity derivatives responsible for the partial exchange of circulation (dot-dash lines) between the

circulation of the original  $y$  –vortices (solid) to the new  $z$  –vortices. This eventually leaves behind the configuration in Fig. 2 of the left-over original  $y$  –vortices (red) to the new  $z$  –vortices (blue).

Figure 4 shows the late-time energy spectra from Kerr, Phys. Fluids (2013) plotted with Kolmogorov scaling and multiplied by  $k^{5/3}$ , giving a Kolmogorov constant  $C_\eta$  roughly consistent with experiments, demonstrating that there exists a turbulent energy cascade despite this being an extremely non-isotropic flow. To further demonstrate that these late-time realisations are turbulent, Donzis et al (2013) compares vorticity moment statistics from this reconnection data set with several calculations of more traditional forced and decaying isotropic, homogeneous turbulence.

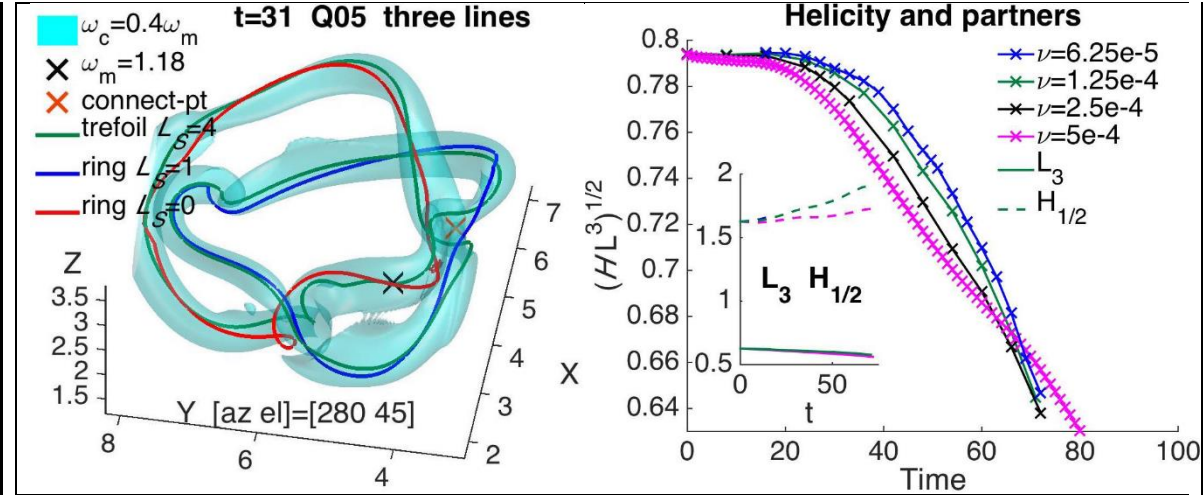
On the other hand, in Kerr, JFM (2013), those same statistics are used to show that if these new initial conditions are evolved under inviscid Euler, there are no singularities. This brings up an ongoing dichotomy between what mathematics and experiments say about finite time events. What experiments say is that all observations are consistent with finite energy dissipation in a finite time. What the mathematics says is that if the Euler evolution of a state is regular and non-singular, then as the viscosity goes to zero a Navier-Stokes solution will be bounded by regular functions of the Euler solutions. Which implies there cannot be finite energy dissipation as the viscosity goes to zero.

The new unpublished Navier-Stokes calculations are at sufficiently high Reynolds number to begin to address these issues. For starters, they show that if the computational box is even larger than the large domains used in Kerr, Phys. Fluids (2013), part of this constraint can be relaxed. But newer mathematics says that initial conditions that either depend upon periodicity, or extend to infinity, can only give partial answer.

Partially for that reason, trefoil vortex knots, one shown in Fig. 5, have been run. The important new global property is the helicity  $H$ , the integral of over space of the alignment of the vorticity and velocity.  $H$  is a global invariant under Euler evolution, and there is an experiment that claims for trefoils,  $H$  is also conserved by the Navier-Stokes equation. In these cases  $H$  is the largest ever generated numerically and in a manner consistent with the experimental visualisation of tracing bubbles suppressed any changes to itself and energy for an extended period. However, beyond the time that the experiments could observe, Fig. 6 shows that the numerical trefoils dissipate finite helicity in a finite time as both the viscosity and radius of the vortices go to zero.

The last statement is critical. The new anti-parallel calculations show that the mathematics that would suppress Navier-Stokes singularities does apply to any fixed initial condition, validating the simulation method. In contrast, the variable radius trefoil calculations show that these bounds can be mitigated by using thinner, more physical vortices. Furthermore, these calculations suggest that the dynamics is controlled by the helicity and its associated self-linking number. A number that is the sum of the traditional Biot-Savart Gauss linking, known as the writhe, and the inherent twist, integral of torsion of the vortex lines.

This work is ongoing, currently generating the further simulations and analysis needed for submission to journals.



5) Trefoil vortex knot beginning to reconnect into red/blue linked rings.

6) Helicity and bounding  $L_3$  and  $H_{1/2}$  norms with finite helicity dissipation for all viscosities.

**7. Names of key academics and any collaborators:**

Robert M. Kerr (Warwick), John Gibbon (imperial)

**8. Sources of significant sponsorship (if applicable):**

**9. Who should we contact for more information?**

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