THE UNIVERSITY OF WARWICK
THIRD YEAR EXAMINATION: APRIL 2010
GEOMETRY OF CURVES AND SURFACES

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

Throughout this paper, \( \mathbb{R} \) will denote the real numbers.

1. Explain how one defines the “tangent”, \( T \), “normal”, \( N \), “binormal”, \( B \), “curvature”, \( \kappa \), and “torsion”, \( \tau \), at a point on a regular unit speed curve, \( \gamma \), in \( \mathbb{R}^3 \), stating when the definitions make sense. [5]

Show that a curve has zero curvature everywhere if and only if it is a straight line. [2]

Show that a curve with nowhere vanishing curvature has zero torsion everywhere if and only if it lies in some plane in \( \mathbb{R}^3 \). [4]

Let \( \gamma \) be a unit speed curve of nowhere vanishing curvature, \( \kappa \), and nowhere vanishing torsion, \( \tau \).

Write \( r(s) = 1/\kappa(s) \) and \( q(s) = 1/\tau(s) \) (where \( s \) is the arc-length parameter), and write \( r'(s) = dr(s)/ds \).

(a) Suppose that \( \gamma \) lies on the unit sphere in \( \mathbb{R}^3 \). Show that \( \gamma = -rN - r'qB \).
[Proceed, for example, by repeatedly differentiating and rearranging the identity \( \gamma(s)\gamma(s) = 1 \).]

Deduce that \( r^2 + (r'q)^2 = 1 \) everywhere on \( \gamma \). [7]

(b) Conversely, suppose that \( r^2 + (r'q)^2 = 1 \) everywhere on \( \gamma \). Show that \( \gamma \) lies on a unit sphere about some point in \( \mathbb{R}^3 \).
[For example, let $\beta = \gamma + r\mathbf{N} + r'q\mathbf{B}$, and show that $\beta' = 0$, so that $\beta$ is constant. You may also find it useful to differentiate the identity $r^2 + (r'q)^2 = 1$.] [7]

2. Define the term “parameterised surface” in $\mathbb{R}^3$. [2]

Explain what is meant by saying that a subset, $S \subseteq \mathbb{R}^3$, is a “regular surface”, defining the terms “chart” and “atlas”. [5]

Show that the unit 2-sphere

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is a regular surface. [4]

If $S$ is a regular surface, and $p \in S$, define the tangent space, $T_p(S)$. Explain why it is well defined, stating any results that you use. [4]

Show that a vector $a \in \mathbb{R}^3$ lies in $T_p(S)$ if and only if there is some $\epsilon > 0$ and a smooth path $\gamma : [-\epsilon, \epsilon] \to \mathbb{R}^3$, with $\gamma([-\epsilon, \epsilon]) \subseteq S$, $\gamma(0) = p$ and $\gamma'(0) = a$. (State clearly any results that you use.) [6]

Show that the cone,

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$$

is not a regular surface. [4]
3. Let $S$ be a regular surface.

Say what it means for a function $f : S \rightarrow \mathbb{R}^n$ to be “smooth”. [2]

Define the terms “smooth (ambient) vector field” and “smooth tangential vector field” on $S$. [2]

If $a$ is a smooth tangential field, and $f : S \rightarrow \mathbb{R}$ is smooth, say how the “covariant (or directional) derivative”, $\nabla_a f = D_a f$ is defined. [3]

If $b$ is any smooth vector field on $S$, say how the “directional derivative”, $D_a b$, and the “covariant derivative”, $\nabla_a b$ are defined. [3]

If $a$, $b$ and $c$ are tangential fields, show that

$$\nabla_a (b . c) = b . \nabla_a c + c . \nabla_a b.$$ [3]

For any tangential fields, $a$, $b$, write $[a, b] = D_a b - D_b a$. [3]

Given a chart, $r$, with co-ordinates, $u, v$, write $a = a_1 r_u + a_2 r_v$, $b = b_1 r_u + b_2 r_v$. [6]

Show that $[a, b]$ can be expressed in terms of $r_u$, $r_v$, the coefficients $a_1, a_2, b_1, b_2$, and their partial derivatives with respect to $u$ and $v$.

Deduce that $[a, b]$ is a tangential field, and that

$$[a, b] = \nabla_a b - \nabla_b a.$$ [2]

Let $n$ be the unit normal field. Show that $b . D_a n = -n . D_a b$ and $a . D_b n = -n . D_b a$. [2]

Deduce that $a . \nabla_b n = b . \nabla_a n$. [2]
4. Let \( r : U \rightarrow \mathbb{R}^3 \) be a parameterised surface. Write \( n \) for the unit normal field.

Explain how the second fundamental form can be defined using the first variation of the first fundamental form of the normal displacement, \( r(t) = r - tn \), of \( r \). (You may assume that \( r(t) : U \rightarrow \mathbb{R}^3 \) is a parameterised surface for small \( t \).) \[4\]

Derive an expression for the second fundamental form in terms of the first derivatives of \( r \) and \( n \) with respect to the co-ordinates, \((u, v)\) in \( U \). Deduce another expression in terms of \( n \) and the second derivatives of \( r \).

Let \( \Theta : T_p(S) \rightarrow T_p(S) \) be the shape operator, where \( p = r(u, v) \) and \( S = r(U) \).

If \( a, b \) are tangential vector fields, explain using the above formulae why

\[
\Pi(a, b) = a \cdot \Theta(b)
\]

is the same as the second fundamental form. \[3\]

Define the principal curvatures, \( \kappa_1, \kappa_2 \), of \( S \). \[2\]

Show that if \( \kappa_1, \kappa_2 \geq 0 \), then for any tangential vector field, \( e \), we have \( |e \cdot \Theta(e)| \geq \min \{\kappa_1, \kappa_2\} \).

Let \( \gamma \) be a unit speed curve in \( S \). Show that \( |\gamma''(s) \cdot n| = |\gamma'(s) \cdot \Theta(\gamma'(s))| \) and deduce that the curvature of \( \gamma \) is at least \( \min \{\kappa_1, \kappa_2\} \) at each point \( \gamma(s) \). \[5\]

5. Let \( r : U \rightarrow \mathbb{R}^3 \) be a parameterised surface. State how the Gauss curvature, \( \kappa \), and the mean curvature, \( H \), are defined in terms of the shape operator. \[4\]

Derive formulae for \( \kappa \) and \( H \) in terms of the coefficients of the first and second fundamental forms. (You may assume a formula for the shape operator in terms of these coefficients.) \[6\]

State Gauss’s Theorema Egregium regarding the Gauss curvature. Would a similar statement hold if Gauss curvature were replaced by mean curvature? Explain your answer. \[4\]

State what it means for \( r \) to be conformal. \[2\]

If \( r \) is conformal, and \( r_{uu} + r_{vv} = 0 \) everywhere, show that \( r \) is a minimal surface. \[2\]

Now let \( r : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be the catenoid:

\[
r(u, v) = (\cosh u \cos v, \cosh u \sin v, u).
\]

Calculate both fundamental forms and the Gauss curvature. Show that \( r \) is conformal and a minimal surface. \[7\]