ST3180
UNIVERSITY OF WARWICK
THIRD YEAR EXAMINATIONS: SUMMER 2009

PROBABILITY THEORY
Time allowed: 2 hours

Full marks may be obtained by correctly answering three complete questions. Candidates may attempt all questions. Marks will be awarded for the best three answers only. Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

1: (a) Suppose that \( E_1, E_2, \ldots \) form a sequence of events drawn from the probability space \((\Omega, \mathcal{F}, P)\).

(i) Write down the set-theoretic definition of \([E_n \text{ i.o.}]\) ("\(E_n\) occurs infinitely often").

(ii) Write down the set-theoretic definition of \([E_n \text{ ev.}]\) ("\(E_n\) occurs eventually").

(iii) State the two Borel-Cantelli lemmas.

(b) Suppose that \(F_1, F_2, \ldots\) form an independent sequence of \(\sigma\)-algebras, where each \(F_n\) is a sub-\(\sigma\)-algebra of \(\sigma\)-algebra of the probability space \((\Omega, \mathcal{F}, P)\).

(i) Define the tail \(\sigma\)-algebra \(T\) for this sequence of \(\sigma\)-algebras.

(ii) Prove that \(T\) is independent of \(\sigma(F_1, \ldots, F_n)\) for all \(n\).

(iii) Hence deduce the Kolmogorov zero-one law for events in this tail \(\sigma\)-algebra.

(c) Suppose that \(X_1, X_2, \ldots\) are independent identically distributed Bernoulli random variables with \(P[X_m = 1] = P[X_m = 0] = \frac{1}{2}\) for all \(m\). Fix \(\alpha > 0\) and set \(k_n = [2^\alpha + 2^{2\alpha} + \ldots + 2^{n\alpha}]\). (Here \([x]\) is the largest integer less than or equal to \(x\).) Let \(A_n\) be the event that the finite sequence \(X_{k_n-1+1}, \ldots, X_{k_n}\) exhibits a run of \(n\) consecutive 1’s: \(X_{r+1} = \ldots = X_{r+n} = 1\) for some integer \(r\) with \(k_n-1 < r \leq r + n \leq k_n\). Show that \(P[A_n \text{ i.o.}]\) is zero or one, and demonstrate that \(P[A_n \text{ i.o.}] = 0\) if \(\alpha < 1\).

(d) In the above, show that \(P[A_n \text{ i.o.}] = 1\) if \(\alpha \geq 1\), by considering events of the form \(B_{n,r} = [X_{k_n-1+(r-1)n+1} = \ldots = X_{k_n-1+rn} = 1]\).

2: (a) Write brief notes to define the following concepts:

(i) Convergence almost surely.

(ii) Convergence in \(p\)-norm.

(iii) Uniform integrability.

(b) Suppose that a sequence of random variables \(X_1, X_2, \ldots\) is such that \(E[X_n^2] < 1\) for all \(n\). Show that the sequence is uniformly integrable.

(c) Suppose that \(U\) is a random variable which is uniformly distributed over \([0,1]\). For each of the following sequences of random variables, say whether it converges almost surely, whether it converges in \(1\)-norm, and whether it is uniformly integrable, giving brief reasons for your answers.
(i) $X_n = U$ if $n$ is odd, $X_n = -U$ if $n$ is even.
(ii) $X_n = n$ if $U \leq 1/n$, $X_n = 0$ if $U > 1/n$.
(iii) $X_n = \sqrt{n}$ if $U \leq 1/n$, $X_n = 0$ if $U > 1/n$.

(d) Suppose the random variables $H_i$ are indicator random variables of a sequence of independent trials each of success probability $1/2$, so that $P[H_i = 1] = 1 - P[H_i = 0] = 1/2$ and the $H_i$ are independent. Set

$$L_n(\theta) = \prod_{i=1}^{n} \theta^{H_i} (1 - \theta)^{1 - H_i},$$

for fixed $\theta$ in $(0, 1)$. (Thus $L_n(\theta)$ is the likelihood that the success probability equals $\theta$ after $n$ trials have been observed.) Show that $L_n(\theta)$ always converges in 2-norm and identify the limit.

3: (a) Write brief notes describing the following:

(i) Weak convergence of probability measures on the real line.
(ii) Convergence of distribution functions.
(iii) Convergence in probability.
(iv) The relationship between convergence of distribution functions and weak convergence on the real line.
(v) The relationship between convergence of distribution functions and convergence in probability.

(b) Prove that convergence of distribution functions is implied by weak convergence of probability measures. You need only consider the special case of probability measures on the real line.

(c) In each of the following cases explain whether $X_n$ converges in the sense of convergence of distribution functions, and if so then what is the limiting distribution, and give reasons for your answers.

(i) For $X$ uniformly distributed on $[-1, 1)$, set $X_n = k/n$ when $k/n \leq X < (k+1)/n$, for $k = -n, \ldots, n-1$.
(ii) $P[X_n = \frac{k}{n}] = 1 - \frac{1}{n}, P[X_n = 1] = \frac{1}{2n}, P[X_n = -1] = \frac{1}{2n}$.
(iii) $P[X_n = \frac{k}{n}] = \frac{3}{4}, P[X_n = n - \frac{1}{n}] = \frac{1}{4}$.
(iv) For $X$ uniformly distributed on $[0, 1)$, and $Y_n = k/n$ when $k/n \leq X < (k+1)/n$, for $k = 0, \ldots, n-1$, set $X_n = \exp(-Y_n)$.

(d) Suppose that $Z_1$, $Z_2$, $\ldots$ are independent and identically distributed, with common distribution being exponential of unit mean. Show that

$$X_n = \max\{Z_1, \ldots, Z_n\} - \log n$$

converges in the sense of convergence of distribution functions, and determine the distribution function of its limit.
4: (a) Suppose that \( X \) is a (real-valued) random variable based on the probability space \((\Omega, \mathcal{F}, P)\), such that its expectation \( \mathbb{E}[X] \) is finite. Let \( G \subseteq \mathcal{F} \) be a sub-\( \sigma \)-algebra.

(i) Define the conditional expectation \( \mathbb{E}[X|G] \).

(ii) Explain what is meant by saying that the random process \( \{X_n : n \geq 0\} \) is adapted to the filtration \( \{\mathcal{F}_n : n \geq 0\} \) of \( \sigma \)-algebras.

(iii) What conditions must hold if \( \{X_n : n \geq 0\} \) is to be a martingale and \( T \) is to be a stopping time with respect to this filtration?

(b) Suppose that the random process \( \{X_n : n \geq 0\} \) is adapted to the filtration \( \{\mathcal{F}_n : n \geq 0\} \) of \( \sigma \)-algebras. For a fixed Borel-measurable set \( B \), show that the first hitting time \( T = \inf\{k \geq 0 : X_k \in B\} \) is a stopping time.

(c) Suppose that \( Y_1, Y_2, ..., Y_m \) are independent and identically random variables of finite mean \( \mu \).

(i) Prove that \( \mathbb{E}[Y_1|Y_1 + .. + Y_n] = (Y_1 + ... + Y_n)/n \).

(ii) Hence show that if \( X_n = (Y_1 + ... + Y_n)/n \) then \( \mathbb{E}[X_n|X_{n+1}] = X_{n+1} \).

(iii) Hence show that \( \{X_{m-n} : n = 0, ..., m-1\} \) is a martingale with respect to the filtration \( \{\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_{m-1}\} \), where \( \mathcal{F}_n = \sigma\{X_{m-n}, X_{m-n+1}, ..., X_m\} \).

(d) A gambling friend offers you the following game. The friend secretly records a sequence of ten independent and identically distributed unbiased coin tosses. Set \( Y_i = 1 \) if the \( i \)th coin toss is heads, \( Y_i = 0 \) otherwise, so \( \mathbb{P}[Y_i = 1] = \frac{1}{2} \). The friend recites the sequence of running averages of the \( Y_i \) in reverse order: \( (Y_1 + ... + Y_{10})/10, (Y_1 + ... + Y_9)/9, ..., (Y_1 + Y_2)/2, Y_1 \). At any stage you may stop the recitation: if the running score is \( (Y_1 + ... + Y_k)/k \) then with conditional probability \( (Y_1 + ... + Y_k)/k \) your friend buys you a beer. (In particular, if you wait till the very end then you are bought a beer if \( Y_1 = 1 \).) You are interested in the absolute probability of being bought a beer, and how it depends on your strategy.

(i) Explain how to use martingale theory to understand how your strategy affects this absolute probability.

(ii) Calculate the probability of ending up being bought a beer if you adopt the best strategy.

(iii) How does your answer change if the coins are independent and identically distributed but biased?