THE UNIVERSITY OF WARWICK
THIRD YEAR EXAMINATION: JUNE 2010
COMPLEX ANALYSIS

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

1. a) State Taylor’s theorem for (complex) analytic functions.
   b) Let $f$ be an analytic function on a domain $D$ and suppose that the set

   $$ Z = \{ z \in D : f(z) = 0 \} $$

   has a limit point $a \in D$. Show that $f^{(n)}(a) = 0$ for all $n \geq 0$.
   c) Show that the function $f$ of part (b) is identically zero on $D$.
   d) Let $f$ be an analytic function on $\mathbb{C}\setminus\{0\}$ which satisfies

   $$ |f(z)| \leq |\log |z||, \quad z \neq 0. $$

   Show that $f(z) \equiv 0$.

2. a) State the homotopic version of Cauchy’s theorem.
   b) State Morera’s theorem.
   c) Show that, if $f$ has an isolated singularity at a point $a$ and $\lim_{z \to a}(z-a)f(z) = 0$, then $f$ has a removable singularity at $a$.
   d) State Liouville’s theorem.
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Question 3 continued

e) Let $f$ be a non-constant entire function without zeros. Show that there is a sequence $z_n$ such that $|z_n| \to \infty$ and $f(z_n) \to 0$. [7]

3.  
   a) State the Maximum Modulus Theorem. [4]
   
b) Let $f$ be continuous on $|z| \leq 1$ and analytic on $|z| < 1$. Suppose $|f(z)| = 1$ for $|z| = 1$ and $|f(0)| < 1$. Show that $f(a) = 0$ for some $a$ with $|a| < 1$. [6]
   
c) Let $|\alpha| < 1$. By considering the Möbius map
   $$T(z) = \frac{z - \alpha}{1 - \bar{\alpha}z},$$
   or otherwise, show that somewhere in the open disk $|z| < 1$, the function $f$ of part (b) takes the value $\alpha$. [6]
   
d) State the open mapping theorem. [4]
   
e) Is there a one-to-one analytic map from the punctured disk $0 < |z| < 1$ onto the annulus $r < |z| < R$, where $r > 0$? Briefly justify your answer. [5]

4.  
   a) State the Argument Principle. [5]
   
b) State and prove Rouché’s theorem. [10]
   
c) Determine explicitly the largest disk centered at the origin on which the function $z^2 - z$ is one-to-one. [10]

5.  
   a) Define residue. [3]
   
b) State Cauchy’s residue theorem. [5]
   
c) Let $a \in \mathbb{C}\setminus\{0\}$. Find the poles of the function
   $$f(z) = \frac{z^4 + 1}{z^2(z - a)(z - 1/a)}$$
   in $\mathbb{C}$ and compute their residues. [8]
   
d) Suppose $0 < |a| < 1$. Compute
   $$\int_0^{\pi} \frac{\cos(2t)dt}{1 - 2a \cos t + a^2}.$$ [9]