MA 3F20

THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: JUNE 2010

KNOT THEORY

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

1. a) Define shadows and their regions.
   b) Define diagrams and alternating diagrams.
   c) Define a two-coloring of a shadow $D$.
   d) Define the parity of a region $R$ of a shadow $D$.
   e) Prove that the parity is well defined and adjacent regions have opposite parity; hence, two-colorings exist.
   f) Prove that every shadow represents exactly two alternating diagrams.

2.  a) Describe the Reidemeister moves.
    b) Define the notion of a coloring of a diagram mod $n$. What does it mean for a
    coloring to be non-trivial?  
    c) Suppose that $D$ and $D'$ are diagrams related by an $R_2$ move. Prove that $D$
    has a non-trivial coloring mod $n$ if and only if $D'$ does as well.
    d) Define the coloring group $Col(L)$ of a link $L$.
    e) For the diagram $D$ shown find the coloring group $Col(D)$. Hence or otherwise
    compute the set
    \[
    \{ n > 1 \mid D \text{ has a non-trivial coloring mod } n \}.
    \]

3.  a) Define the braid group $B_n$ by giving generators and relations.
    b) Define the Markov moves on braids.
    c) Define the closure of a braid.
    d) Briefly discuss the connection between braid relations and Markov moves on
    the one hand and the Reidemeister moves on the other hand.
    e) Draw and identify the closures of the braids $(\sigma_1\sigma_3)^3 \in B_4$, $(\sigma_1\sigma_2)^2 \in B_3$, and
    $\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2 \in B_4$.
    f) Suppose that $\sigma = \sigma_1\sigma_3 \in B_3$ and $\tau = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} \in B_4$. Prove
    that there is a sequence of Markov moves connecting $\sigma$ and $\tau$. State clearly
    theorems from class used, if any.
4.  a) Define the writhe \( w(D) \) of an oriented diagram \( D \). Define the linking number \( \text{lk}(L) \) of a two-component link \( L \).  

b) Given a diagram \( D \) for an oriented knot \( K \) show that there is a two-component link \( L \) with \( w(K) = \text{lk}(L) \), with \( K \subset L \), and with the other component close to \( K \).  

c) Give the axioms for the Kauffman bracket of an oriented diagram \( D \). Define the Kauffman polynomial of a link \( L \).  

d) Define the Jones polynomial of an oriented link. Calculate the highest and lowest degree term of the Jones polynomial of the indicated knot.

5.  a) Define a spanning surface for a knot \( K \).  

b) Define a Seifert surface for a knot \( K \). Define \( g(K) \), the genus of \( K \).  

c) Suppose that \( D \) is a diagram for \( K \). Demonstrate a relationship between the genus of \( K \) and the number of Seifert circles and crossings of \( D \).  

d) Compute the genus of the knot shown below. Carefully state any theorems from class that you use.