MA3G0

THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: JUNE 2010

MODERN CONTROL THEORY

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

1. Consider the system

\[ \dot{x} = Ax + Bu \]
\[ x(0) = x_0 \in \mathbb{R}^n, \]

with \( u \in U = \{ u : [0, \infty) \rightarrow [-1, 1]^m : u \text{ is measurable} \} \), \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \).

Let the target be \( 0 \in \mathbb{R}^n \).

a) Define the controllable set \( C \). \hspace{1cm} [2]

b) Show that if \( \text{rank}[B, AB, \ldots, A^{n-1}B] < n \), then there exists a fixed hyperplane in \( \mathbb{R}^n \) which contains \( C \). \hspace{1cm} [7]

c) Assume that \( \text{rank}[B, AB, \ldots, A^{n-1}B] = n \) and \( A \) has a real eigenvalue \( \lambda_1 > 0 \).

Prove that there exists \( x_0 \in \mathbb{R}^n \) such that \( x_0 \notin C \). \hspace{1cm} [7]

d) Let \( A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \), \( B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \).

i) Show that \( C \subset \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = x_2 \in (-2, 2)\} \). \hspace{1cm} [5]

ii) Let \( x_0 \in \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = x_2 \in (-2, 2)\} \). Find a control function \( u \in U \) which steers \( x_0 \) to 0 and therefore conclude that \( C = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = x_2 \in (-2, 2)\} \). \hspace{1cm} [4]
2. a) Consider the system
\[ \dot{x} = f(x,u), \quad x(0) = x_0 \in \mathbb{R}^n \]  
with \( u : [0,\infty) \to U \subset \mathbb{R}^m \) measurable, \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) continuously differentiable and \( f(0,0) = 0 \). Let \( A_f = f_x(0,0) \) and \( B_f = f_u(0,0) \) and assume that \( \dot{x} = A_f x + B_f u \) is controllable. (In your answer to the following questions state clearly any results that you assume).

i) Let \( U = [-1,1]^m \) and show that if for \( u \equiv 0 \) the system (1) is globally asymptotically stable, then the controllable set \( C = \mathbb{R}^n \). [7]

ii) Let \( U = \mathbb{R}^m \) and prove that there exists a matrix \( F \in \mathbb{R}^{m \times n} \) such that \( \dot{x} = f(x,Fx) \) is locally asymptotically stable at \( x = 0 \). [10]

b) Consider the system
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = g(x_1,x_2) + h(u), \]
with \( u : (0,\infty) \to [-1,1] \). Assume that \( g : \mathbb{R}^2 \to \mathbb{R} \) and \( h : [-1,1] \to \mathbb{R} \) are continuously differentiable with \( g(0,0) = 0 \) and \( h(0) = 0 \).

i) Show that the system is locally controllable when \( h(u) = u \). [3]

ii) Let \( h : [-1,1] \to [-1,1] \) be bijective and prove that the system is still locally controllable. [5]
3. a) Consider the system
\[ \dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n \quad (2) \]
with \( f : \mathbb{R}^n \to \mathbb{R}^n \) continuously differentiable and \( f(0) = 0 \). Let \( G \subset \mathbb{R}^n \) be a neighbourhood of \( x = 0 \) and \( V : G \to \mathbb{R} \) be continuously differentiable. Which conditions make the function \( V \) a strict Liapunov function? [3]

b) Assume that all eigenvalues of \( A_f = f_x(0) \) have negative real parts. Show that the system (2) is locally asymptotically stable at \( x = 0 \) (state clearly any results that you use). [10]

c) Let \( A \in \mathbb{R}^{n \times n} \) and \( \sigma \in \mathbb{R} \). Assume that there exists a positive definite matrix \( P \in \mathbb{R}^{n \times n} \) such that
\[ A^T P + PA + 2\sigma P = -I. \]
Show that all eigenvalues of \( A \) have real parts less than \(-\sigma\). [7]

d) Consider the system described by
\[ \begin{align*}
\dot{x}_1 &= e^{x_2} + x_1 u_1, \\
\dot{x}_2 &= x_2 u_2,
\end{align*} \]
with \( u_1, u_2 : [0, \infty) \to \mathbb{R} \) measurable. Show that for an appropriate choice of \( u_1 \) and \( u_2 \) the system is asymptotically stable at \( x = 0 \). (Hint: Choose \( V(x_1, x_2) = x_1^2 + x_2^2 \).) [5]
4. a) Consider the system

\[ \dot{x} = Ax + Bu, \quad x(0) = x_0 \in \mathbb{R}^n \]  

(3)

with \( u : [0, \infty) \to [-1, 1]^m \) measurable, \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) and the target \( 0 \in \mathbb{R}^n \). Assume that \( x_0 \) is in the controllable set \( \mathcal{C} \) and let \( u^* \) be a time-optimal control with the optimal time \( \tau^* \).

i) Define the reachable set \( K(t, x_0) \).

ii) Prove that \( 0 \in \partial K(\tau^*, x_0) \) (state carefully any results that you assume).

iii) Which condition makes the system (3) normal?

iv) Prove that if (3) is normal then \( u^* \) is unique.

b) Consider the system

\[ \dot{x}_1 = x_1 + u, \quad \dot{x}_2 = -x_1. \]

with \( u(t) \in [-k, k], \ k > 0 \). Use the maximum principle to show that the time-optimal control (when it exists) has at most one switch.
5. a) Let \( x \) satisfy
\[
\dot{x} = f(x, u), \quad x(0) = x_0 \in \mathbb{R}^n
\]
with \( u : [-1, 1]^m \to \mathbb{R}^n \) measurable and \( f \) continuously differentiable in \( x \) and \( u \). Consider the problem of finding the control \( u \) which steers \( x_0 \) to the origin at time \( T \) while minimizing the cost functional
\[
J(u) = \int_0^T r(x(s), u(s)) \, ds,
\]
where \( r \) is continuous in \( u \) and continuously differentiable in \( x \).

i) Define the extended costate \( \hat{p}(t) \in \mathbb{R}^{n+1} \) for the above problem.

ii) Let \( H(\hat{p}, x, u) = \hat{p}_{n+1} r(x, u) + \sum_{j=1}^n \hat{p}_j f_j(x, u) \). Assume that for any \( x_0 \) in some open neighbourhood \( N \) of the origin, an optimal control \( u^* \) with the corresponding response \( x^*(\cdot) \) exists. Suppose that \( J(u^*(t, x_0)) \) is differentiable with respect to \( x_0 \). Show that there exists \( \hat{p} \in \mathbb{R}^{n+1} \) such that \( H(\hat{p}, x^*, v) \leq 0 \) for any \( v \in [-1, 1]^m \); and \( H(\hat{p}, x^*, u^*) = 0 \).

b) Consider the system
\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = u,
\]
with \( u(t) \in [-1, 1] \). Show that the control \( u^* \) which minimizes
\[
\int_0^T (|u(s)| + 1) \, ds,
\]
(when it exists) can only take the three values -1, 0 and 1 (assume that \( \hat{p}_3 < 0 \)). Write down the equations governing the costates and find out the maximum number of times that \( u^* \) can change its value.