a) Define what it means for a sequence \((a_n)\) to converge to a limit \(a\).

Define what it means for the sequence to be bounded.

Prove that every convergent series is bounded. \hfill [6]

b) Which of the following sequences converge? Justify your answers. Find the limit for those sequences that converge to a real number.

(i) \(a_n = \frac{\sin(n) + \cos(n)}{\log(n)}, \text{ for } n \geq 2\)

(ii) \(b_n = \frac{\log(n)}{3 + \sin(n) + \cos(n)}, \text{ for } n \geq 2\)

(iii) \(c_n = \frac{\sin(n) + 2\log(n)}{\cos(n) + \log(n)}, \text{ for } n \geq 2\) \hfill [6]

c) Given a sequence \((a_n)\), recall that the \(n\)th term \(a_n\) is called a floor term if for every \(m > n\) we have that \(a_m \geq a_n\).

Using this notion, show that any sequence which is bounded below has either an increasing subsequence or a strictly decreasing subsequence. \hfill [8]
Determine whether the following statements are true or false. If true, give a brief justification; if false, give a counterexample. (You will get zero marks if you do not provide either a justification or a counterexample.) You will get one point if provide full and correct answers for all 8 questions.

a) If $A$ is an orthogonal $n \times n$ matrix, then so is $A^{-1}$.  

b) Let $n \geq 2$. If $A$ and $B$ are $n \times n$ matrices with $\text{rank}(A) = \text{rank}(B)$, then $\text{rank}(AB) = \text{rank}(A) = \text{rank}(B)$.  

c) Let $n \geq 2$. If $A$ and $B$ are $n \times n$ matrices such that their characteristic polynomials coincide, then $A$ and $B$ are similar.  

d) If $V$ is a vector space over $\mathbb{R}$ spanned by vectors $v_1$, $v_2$ and $v_3$, then $\dim_{\mathbb{R}} V = 3$.  

e) The map $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T(x, y, z) = (-x - y, 5 - z, y + 3)$ is linear.  

f) If $A$ is a $2 \times 2$ real matrix without real eigenvalues, then $A$ is a rotation matrix.  

g) If $U$, $W$ and $Z$ are subspaces of a vector space $V$, then $U \cap (W + Z) = (U \cap W) + (U \cap Z)$.  

h) If $I_V$ is an identity operator on a vector space $V$, $\dim V = n$ and $A$ is a matrix corresponding to $I_V$, then $A = I_n$.  

a) Let \( A, B \) be two distinct points in the plane. Let \( c \) respectively \( d \) be a circle around \( A \) respectively \( B \) with radius \( AB \). Let \( P \) and \( Q \) be the two intersection points of the circles \( c \) and \( d \). Let \( X \) be the intersection point of the line \( PQ \) and the line \( AB \).

(i) Show that \( AQP = BQP \).  
(ii) Let \( Y \) be any point on the line \( PQ \). Show that \( AY = BY \).  
(iii) Show that \( QXB \) and \( QXA \) are right angles. The line through \( P \) and \( Q \) will henceforth be called the **perpendicular bisector of** \( AB \).  

b) Let \( K \) and \( L \) be parallel lines in the plane (i.e. lines which do not intersect). Let \( M \) be a third line which intersects \( K \) at an angle \( \alpha \). What can you say about the relationship between \( L \) and \( M \)? No justification is required.  

c) Let \( ABC \) be a planar triangle. Let \( K \) be the perpendicular bisector of \( BC \) and let \( L \) be the perpendicular bisector of \( AC \). Show that \( K \) and \( L \) intersect.  

d) Let \( ABC \) be a planar triangle. Show that there exists a circle which goes through \( A, B \) and \( C \).