THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: APRIL 2010

ADVANCED PDE

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.
1. a) Let $U$ be an open subset of $\mathbb{R}$ and let $u, v \in L^1_{\text{loc}}(U)$. Explain what it means for $v \in L^1_{\text{loc}}(U)$ to be the weak derivative of $u$. 

b) Define $H : (0, 2) \to \mathbb{R}$ by

$$H(x) := \begin{cases} 
  x, & \text{if } x \in (0, 1], \\
  1, & \text{if } x \in (1, 2).
\end{cases}$$

Compute the weak derivative of $H$. 

c) Define $H : \mathbb{R} \to \mathbb{R}$ by

$$H(x) := \begin{cases} 
  1, & \text{if } x \geq 0, \\
  -1, & \text{if } x < 0.
\end{cases}$$

Prove that $H$ is not weakly differentiable.
2. Let \( U \) be a bounded, connected and open subset of \( \mathbb{R}^n \) with a \( C^1 \) boundary. Let \( u \in W^{1,p}(U) \), \( p \in [1, \infty) \). Let \( \bar{f}_U u \) denote the average of \( u \), that is the integral of \( u \) over \( U \) divided by the measure of \( U \).

a) Show that if \( Du = 0 \) almost everywhere in \( U \) then \( u \) is constant almost everywhere in \( U \).

(Hint: use mollifications of \( u \) and their properties.)

b) Show that \( \|u - \bar{f}_U u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)} \), \( C \) independent of \( u \).

(Hint: arguing by contradiction there exists a sequence of functions \( u_k \) such that \( \|u_k - \bar{f}_U u_k\|_{L^p(U)} > k\|Du\|_{L^p(U)} \). Define \( v_k := \frac{u_k - \bar{f}_U u_k}{\|u_k - \bar{f}_U u_k\|_{L^p(U)}} \).

(i) Compute \( \bar{f}_U v_k \), \( \|v_k\|_{L^p(U)} \) and \( \|Dv_k\|_{L^p(U)} \).

(ii) Use the fact that \( W^{1,p}(U) \subset L^p(U) \) and part a) to show that \( v_k \) converges to a constant in \( L^p(U) \).

(iii) Derive a contradiction.

Note: in order to solve part b) you are allowed to assume part a) even if you were not able to solve it.
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3. Let $U$ be a bounded, connected and open subset of $\mathbb{R}^2$ with a $C^1$ boundary.

a) Explain what it means for $u$ to be in $W_0^{1,p}(U)$. [5]

b) Show that if $u \in C_0^\infty(\mathbb{R}^2)$ then $\int_{\mathbb{R}^2} u^2 \leq (\int_{\mathbb{R}^2} |Du|)^2$. [12]

(Hint: define $I_u := \{ y \in \mathbb{R} \mid (x, y) \in U \}$ and let $M_x u(x) := \max_{y \in I_x} u(x, y)$. Begin by showing that $M_x u(x) \leq \int_{-\infty}^{\infty} |\frac{\partial}{\partial y} u(x, y)| \, dy$.)

c) Use part b) and the definition of $W_0^{1,2}(U)$ to conclude that for any $u \in W_0^{1,2}(U)$, $\int\int_U u^2 \leq (\int\int_U |Du|)^2$. [8]

Note: in order to solve part c) you are allowed to assume part b) even if you were not able to solve it.
4. Let $U$ be a bounded, connected and open subset of $\mathbb{R}^n$ and let $u \in W^{1,1}(U)$.

a) Prove that $|u| \in W^{1,1}(U)$ and that

$$D|u|(x) = \begin{cases} 
Du(x) & \text{a.e. on } E_+ := \{x \in U : u(x) > 0\}, \\
-Du(x) & \text{a.e. on } E_- := \{x \in U : u(x) < 0\}, \\
0 & \text{a.e. on } E_0 := \{x \in U : u(x) = 0\}.
\end{cases}$$

(Hint: $|u(x)| = \lim_{\varepsilon \to 0} F_\varepsilon(u(x))$, where $F_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2}$.)

b) Define $u^+(x) = \max\{0, u(x)\}$. Prove that $Du^+(x) = 0$ a.e. on $E_- \cup E_0$ and deduce that $Du = 0$ a.e. on $E_0$.

(Hint: $u^+ = |u^+|$ and $u^+(x) = \frac{1}{2}(u(x) + |u(x)|)$.

) [8]

c) Prove that $\int_U |D|u|| \, dx = \int_U |Du| \, dx$. [5]

Note: in order to solve part b) you are allowed to assume part a) even if you were not able to solve it. In order to solve part c) you are allowed to assume part a) and b) even if you were not able to solve them.
5. Let $U$ be a bounded, connected and open subset of $\mathbb{R}^n$ with a $C^2$ boundary. Let $\overline{U}$ denote the closure of $U$ in $\mathbb{R}^n$ and let $\nu$ denote the outer pointing unit vector normal to $\partial U$. Recall that since $\partial U$ is $C^2$, it satisfies the interior ball condition at each point. Let $D_1(0)$ denote the disc in $\mathbb{R}^2$ of radius one and centered at the origin and let $\overline{D_1(0)}$ denote the closure of $D_1(0)$ in $\mathbb{R}^2$.

a) Let $a_{ij}$ be a $2 \times 2$ matrix and consider the P.D.E. given by the equation
\[ \sum_{i,j=1}^{2} a_{ij} \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} u = 0. \]
For which values of $t$ would the following matrix give an elliptic PDE? (Justify your answer)
\[ a_{ij} = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}. \] [5]

b) Find all solutions to
\[ \begin{cases} -\Delta u = 0 & \text{in } U, \\ \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial U. \end{cases} \] in $C^\infty(\overline{U})$. [8]

c) Let $u \in C^\infty(\overline{D_1(0)})$ satisfy $-\Delta u \leq -(x^2 + y^2)$.

(i) Show that $u$ does not achieve its maximum in $D_1(0)$. [4]

(ii) Show that $u(0) \leq \sup_{\partial D_1(0)} u - \frac{1}{24}$. [8]

(Hint: Compute $\Delta \left( \frac{x^4 + y^4}{12} \right)$ and use the fact that $\min_{\partial D_1(0)} \frac{x^4 + y^4}{12} = \frac{1}{24}$)

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