1. Let $f(T)$ be a non-constant rational function in one variable over the complex numbers. Suppose that $p$ is a complex number, let $m_p$ be the limiting value as $\epsilon$ tends to zero of the number of connected components of $\{z \in \mathbb{C} : 0 < |f(z) - p| < \epsilon\}$. We say $f$ is totally ramified at $p$ if $m_p = 1$.

   a) Show that $m_p$ is the number of points in the Riemann sphere mapping to $p$ under $f$. [5]

   b) Assume $m_p = 1$ and let $q = a/b$ be the unique point in the Riemann sphere so that $f(q) = p$ (with $b$ possibly zero). Let $h$ and $g$ be the rational functions

   $$h(z) = 1/(z - p), \quad g(z) = (az + c)/(bz + d).$$

   Prove that the rational function $h \circ f \circ g$ is a polynomial for any complex numbers $c, d$. [10]

   c) Prove that the set of rational functions of degree $n$ which are totally ramified at $p \in \mathbb{C}$ is the set of functions of the form $f(z) = p + \frac{(-bz+c)^n}{Q(dz-c,-bz+c)}$ for $ad - bc = 1$ and $Q$ homogeneous of degree $n$. [10]
2. Let \( f(x, y) \) be an irreducible complex polynomial in two variables of total degree two. Suppose that the coefficient of \( x^2 \) is not zero.

   a) Prove that \( x \) is integral over \( \mathbb{C}[y] \) in the ring of polynomial functions on the zero set of \( f \).

   b) Suppose further that the zero set of \( df \wedge dy \) meets the zero set of \( f \) transversely at two points. Prove that there are finite Fourier series \( x(z) \) and \( y(z) \) such that the function
   \[
   \{ z \in \mathbb{C} : 0 \leq \text{Im}(z) < 2\pi \} \to \mathbb{C}^2
   \[ x \mapsto (x(z), y(z)) \]
   is a bijection onto the zero set of \( f \).  

3. Suppose \( f \) and \( h \) are non-constant holomorphic functions on \( \mathbb{C}^2 \).

   a) Give an example to show that the restriction of \( h \) to the zero set of \( f \) need not always be proper. You do not need to justify your example.

   b) Suppose the zero set of \( f \) meets the zero set of \( df \wedge dh \) in the generic way, i.e., with at worst simple crossings. Prove that the inequality
   \[
   2g - 2 \leq d - m + m_t
   \]
   holds, where \( g \) is the genus of the curve \( f = 0 \), \( m \) is the number of simultaneous solutions of \( f = 0, df \wedge dh = 0 \) and \( m_t \) is the limiting number of connected components of the set of \( (x, y) \) with \( f(x, y) = 0, |h(x, y)| > B \) as \( B \) tends to infinity.

   c) Prove that the inequality (*) is an equality if the two coordinates \( x \) and \( y \) are integral over \( \mathbb{C}[h] \) in the ring of polynomial functions on the zero set of \( f \).

4. The Gaussian integers \( \mathbb{Z}[i] \) are the complex numbers of the form \( m + ni \) for \( m, n \in \mathbb{Z} \). Let \( f \) and \( g \) be two meromorphic function of one variable which are holomorphic at all points of \( \mathbb{C} \setminus \mathbb{Z}[i] \), and satisfy \( f(z+1) = f(z) \) and \( f(z+i) = f(z) \) and also likewise \( g(z+1) = g(z) \) and \( g(z+i) = g(z) \) for all \( z \in \mathbb{C} \).

   a) Prove that if \( f \) has at most simple poles then \( f \) is holomorphic everywhere.

   b) Let \( p, q \) be complex numbers. Suppose that \( g \) has a simple zero at all complex numbers which are congruent to \( p \) or \( q \) modulo \( \mathbb{Z}[i] \) but nowhere else. Prove that either the sum \( p + q \) or the difference \( p - q \) is a Gaussian integer.
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5. Suppose $\tau$ is a complex number with $\text{Im}(\tau) > 0$. Let $E$ be the elliptic curve

$$\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau).$$

In this question you may assume that divisors of degree 3 on $E$ are very ample.

a) Define the theta function $\theta(z, \tau)$.\[5]

b) Let $f$ be a meromorphic function on $E$ with a pole of order at most six at $(1 + \tau)/2$. Prove that there exists a unique quadratic form $Q(x_1, x_2, x_3)$ such that

$$f(z) = Q(\theta(z, \tau)^3, \theta(2z, 2\tau), \theta(z, \tau), \theta(3z, 3\tau))/\theta(z, \tau)^3$$

(hint: it is a dimension count).\[5]

c) Conclude that every such meromorphic function is uniquely expressed as a polynomial in $\theta(2z, 2\tau)/\theta(z, \tau)^2$ and $\theta(3z, 3\tau)/\theta(z, \tau)^3$ of degree at most two.\[5]

d) What are the six coefficients of the polynomial of total degree at most two which produces a meromorphic function which has a double zero at each of $1/6 + \tau/2$ and $5/6 + \tau/2$ and a quadruple pole at $1 + \tau/2$ (hint: consider first the zeroes and poles of $\theta(3z, 3\tau)$).\[10]