1. a) Let $\Sigma = \prod_{n=0}^{\infty} \{1, \ldots, N\}$ be a space of sequences with the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{e(x_n, y_n)}{2^n}$$

where $x = (x_n)_{n=0}^{\infty}, y = (y_n)_{n=0}^{\infty} \in \Sigma$ and

$$e(i, j) = \begin{cases} 1 & \text{if } i \neq j; \\ 0 & \text{if } i = j. \end{cases}$$

Show that $\Sigma$ is compact. [7]

b) Given a $N \times N$ matrix $A$ with entries 0 and 1 define the space $\Sigma_A \subset \Sigma$ and the subshift of finite type: $\Sigma_A \to \Sigma_A$.

Define what it means for $\sigma : \Sigma_A \to \Sigma_A$ to be (topologically) transitive. State without proof how this is characterized in terms of the matrix $A$.

Define what it means for $\sigma : \Sigma_A \to \Sigma_A$ to be (topologically) mixing. State without proof how this is characterized in terms of the matrix $A$. [10]

c) Define what it means for $x$ to be a periodic point of period $n$.

Show that the number of periodic points of period $n$ is given by the trace of the $n$-fold matrix product $A^n$, for $n \geq 1$. [8]
2. a) Define what it means for $T : K \rightarrow K$ to be an expanding map on the unit circle $K = \mathbb{R}/\mathbb{Z}$.
   Let $\sigma : \Sigma \rightarrow \Sigma$ be the fullshift on $N$ symbols on $\Sigma = \prod_{n=0}^{\infty} \{1, \ldots, N\}$. Define what it means for a map $\pi : \Sigma \rightarrow K$ to be a semi-conjugacy. \[6\]
   b) Show that there is a semi-conjugacy between the full shift on 2 symbols and the expanding map $T : K \rightarrow K$ given by $T(x) = 2x \pmod{1}$. \[12\]
   c) State explicitly the periodic points of period $n \geq 1$ for the map $T : K \rightarrow K$ given by $T(x) = 2x \pmod{1}$.
   Let $N(n)$ denote the number of periodic points of period $n$ for this map $T$. We denote the zeta function by the complex function
   \[ \zeta(z) = \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} N(n) \right), \quad z \in \mathbb{C}, \]
   which is well defined when $|z| < \frac{1}{2}$. Show that $\zeta(z)$ extends to the entire complex plane as a rational function. \[7\]

3. Let $T : X \rightarrow X$ be a continuous map on a compact metric space.
   a) Define what it means for $Y \subset X$ to be a $(n, \epsilon)$-spanning set for $T$.
      Define what it means for $Y \subset X$ to be a $(n, \epsilon)$-separated set for $T$. \[6\]
   b) Let $N(n, \epsilon)$ denote the least cardinality of any $(n, \epsilon)$-spanning set. Let $S(n, \epsilon)$ denote the largest cardinality of any $(n, \epsilon)$-separated set. Show that:
      i) $N(n, \epsilon) \leq S(n, \epsilon)$;
      ii) $S(n, \epsilon) \leq N(n, \epsilon/2)$.
      Define the topological entropy $h(T)$ of $T : X \rightarrow X$ in terms of spanning sets.
      Define the topological entropy of $T : X \rightarrow X$ in terms of separating sets.
      Show that these two definitions are equivalent. \[11\]
   c) Let $T : X \rightarrow X$ be a homeomorphism of a compact metric space $D$ with metric $d$. We say that $T$ is an isometry if $d(Tx, Ty) = d(x, y)$, for any $x, y \in X$. Show that $h(T) = 0$. \[8\]

4. a) Define what it means for a homeomorphism of a compact metric space to be minimal. Let $f : K \rightarrow K$ be the rotation of the circle $K = \mathbb{R}/\mathbb{Z}$ defined by $f(x) = x + \alpha \pmod{1}$ where $0 \leq \alpha < 1$. Show that $f$ is minimal if $0 < \alpha < 1$ is irrational. \[8\]
5. Let $A$ be a $2 \times 2$ matrix with integer entries and determinant 1. Assume that $A$ has no eigenvalues with absolute value 1.

a) Define the linear hyperbolic toral automorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$ associated with $A$.  

b) Show that the periodic points for $T : \mathbb{T}^2 \to \mathbb{T}^2$ are precisely those points which have rational coordinates.  

c) Define what it means for a sequence of points $(x_n)_{n=-\infty}^{\infty}$ to be a $\delta$-pseudo orbit for $T$.

Define the Shadowing Property for $T$.  

d) We say that any homeomorphism $S : \mathbb{T}^2 \to \mathbb{T}^2$ is expansive if there exists a constant $\delta > 0$ such that if $d(S^n x, S^n y) < \delta$ for all $n \in \mathbb{Z}$ then $x = y$.  

Show that for an expansive homeomorphism the rate of growth of the periodic points is less than the topological entropy, i.e.,

$$\limsup_{n \to +\infty} \frac{1}{n} \log (\text{Card}\{x : S^n x = x\}) \leq h(S).$$