1. The *SIR* model of infectious disease spread is given by:

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - gI \\
\frac{dR}{dt} &= gI
\end{align*}
\]

Here *S* is the proportion of susceptibles, *I* the proportion infectious, and *R* is the proportion recovered. \(\beta\) and \(g\) are positive constants.

a) Find the maximum proportion of infectious individuals during an epidemic, assuming it begins with very few cases. (Hint: use a similar methodology to finding \(R_\infty\)).

b) For the early stages of the recent influenza pandemic, it was observed that the number of cases doubled every week, while the average number of secondary cases produced per infected individual was 1.4. Assuming the simple SIR model is a reasonable description of this epidemic, use these two pieces of data to find the parameters \(\beta\) and \(g\).
c) Pandemic influenza can be modelled as an SIR disease, but a more accurate prediction requires that adults and children need to be modelled separately (although birth, death and ageing processes can be ignored). The transmission rate between children $\beta_{CC}$ is known to be longer than all other transmission rates, which are assumed to be equal ($\beta_{CA} = \beta_{AC} = \beta_{AA}$). The recovery rate ($g$) is equal in both children and adults. Construct a suitable model and show that when complete vaccination of children (only) is sufficient to prevent an epidemic that:

$$\beta_{CC} > \frac{R_0^2 N_A - R_0 N_A - N_C}{(R_0 - 1) N_C} \beta_{AA}$$

where $N_C$ and $N_A$ are the proportion of individuals that are children and adults respectively.

2. The Nicholson-Bailey equations for the interaction between a host ($H$) and parasitoid ($P$) can be written as:

$$H_{t+1} = r H_t \exp (-a P_t)$$
$$P_{t+1} = \lambda r H_t \left[1 - \exp (-a P_t)\right].$$

a) Show how these equations are derived from a consideration of the continuous time interactions, and hence give a biological meaning of the three parameters $\lambda$, $r$ and $a$.

In a more complex system, parasitoids are known to survive for more than one season, although hosts still only live for one season. For the two competing hypotheses below, formulate an appropriate modification to the Nicholson-Bailey model, calculate the non-trival equilibrium and in the limit where $r$ is slightly bigger than one ($r = 1 + \varepsilon$, where $\varepsilon \ll 1$) show whether or not the non-trival equilibrium is stable. (For simplicity you may assume that $\lambda = a = 1$).

b) Assume that a fraction $p$ of all parasitoids survive from one season to the next.

c) Assume that each parasitoid lives for exactly two seasons. (Note you will need to distinguish between parasitoids that are in their first season and those in their second.)
Question 3 continued

3. An infectious disease can be modelled as $SEIR$, ignoring demographic processes:

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI/N \\
\frac{dE}{dt} &= \beta SI/N - aE \\
\frac{dI}{dt} &= aE - gI \\
\frac{dR}{dt} &= gI
\end{align*}
\]

where $\beta$, $a$ and $g$ are all positive constants, $N$ is the total population size, and $S$, $E$, $I$ and $R$ refer to the number of individuals in the susceptible, exposed, infectious and recovered classes.

a) Define and calculate the basic reproductive ratio, $R_0$; and show that $R_0 > 1$ implies that the infection can invade a naive population. [4]

b) A population of $N$ individuals can be partitioned into $N/2$ households, with 2 people within each household. (You may assume that $N$ is very large). A disease with SEIR dynamics spreads through this population, with transmission rate $\beta$ between individuals within a household and transmission rate $\alpha$ between individuals outside the household.
   (i) Find the expected number of cases generated within a household. [3]
   (ii) Hence (by considering the number of secondary household infected by a household that has just been infected) find the threshold level of $\alpha$ that allows a disease to invade. [5]

c) Extend these results for a population partitioned into $N/3$ households, each of which contains 3 people.
   (i) First list the possible transmission routes (from one infected individual) that lead to either 1, 2 or 3 individuals being infected within the household. [4]
   (ii) Calculate the probabilities associated with each of these outcomes [6]
   (iii) Hence calculate the expected final number of infected individuals in an infected household of size 3, and find the threshold level of $\alpha$ that allows a disease to invade. [3]
   (Note: you will need to explicitly consider all possible transmission routes within a household, and will need to consider the distribution of times each individual is infected for.)

4. The Lotka-Volterra predator-prey model can be written as:

\[
\frac{dV}{dt} = BV - CVP
\]

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Question 4 continued

\[ \frac{dP}{dt} = AVP - DP \]

where \( A, B, C \) and \( D \) are all positive constants, and \( V \) and \( P \) refer to the number of prey and predator respectively.

a) Find the non-trivial equilibrium of this system and show that this point is neutrally stable. \([4]\)

b) Construct a comparable model with two predator and one prey species, where the two predator species only interact through their consumption of the same prey species. Show that both predator species cannot generally co-exist, and find the conditions for predator species 1 to out compete predator species 2. \([6]\)

c) Construct a comparable model with one predator and two prey species, where the two prey species only interact through their consumptions by the same predator species. Show that both prey species cannot generally co-exist, and find the conditions for prey species 1 to out compete prey species 2. \([6]\)

d) Construct a similar model for two predator species and two prey species, again assuming that the only interactions arise due to consumption. Show that a non-trivial equilibrium exists for this system, but that this equilibrium point cannot be stable. \([9]\)

5. A sexually-transmitted infection can be described by the SIS model with frequency dependent transmission:

\[ \frac{dS}{dt} = -\beta SI + gI \]
\[ \frac{dI}{dt} = \beta SI - gI \]

where \( \beta \) and \( g \) are positive constants, and \( S \) and \( I \) refer to proportions of the population that are Susceptible and Infectious respectively.

a) Show that the time dependent solution can be written as:

\[ I(t) = \frac{Ae^{rt}}{1 + Be^{rt}} \]

and find the parameters \( A, B \) and \( r \) in terms of the transmission rate \( \beta \) and the recovery rate \( g \) of the infections \([5]\)
b) Two sexually-transmitted infections with SIS dynamics are circulating with a population. It can be assumed that: transmission from an individual infected with both infections always generates another case with both infections; and that recovery always occurs due to treatment which clears all infections. Write down a suitable set of equations to describe the dynamics of these two diseases and explain all your parameters; make sure that you account for the potential of partial immunity or enhanced susceptibility of infected individuals. [7]

c) Use the model defined in part (b), and assume that all basic transmission and recovery rates are equal, and that the two infections generate the same level partial immunity or enhanced susceptibility. Describe the possible dynamics of the two infections by considering the non-trivial equilibrium and the potential for invasion. [13]