The University of Warwick
Fourth Year Examination: April 2010
Lie Algebras

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Answer 4 Questions.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

Notice: All vector spaces and Lie algebras in this exam are assumed to be finite dimensional and defined over \( \mathbb{C} \).

1. Let \( L \) be a Lie algebra over \( \mathbb{C} \), and let \( M, N \) and \( I \) be ideals of \( L \).

   a) Show that there exists a unique solvable ideal \( R(L) \) of \( L \) which contains every solvable ideal of \( L \) (you are allowed to use the Isomorphism Theorems and the fact that if \( M/N \) and \( N \) are solvable, then so is \( M \)). [4]

   b) Explain what it means for \( L \) to be semisimple. [1]

   c) Give a definition of the Killing form \( k \) of \( L \). [1]

   d) Show that \( k \) is a symmetric bilinear form. [2]

   e) Show that the restriction of \( k \) to \( I \) coincides with the Killing form \( k_I \) on \( I \) when \( I \) is considered as an independent Lie algebra, i.e., for all \( x, y \in I \), \( k(x, y) = k_I(x, y) \). [5]

   f) Define \( I^\perp \) with respect to the Killing form \( k \), and show that \( I^\perp \) is an ideal of \( L \). [4]

   g) State Cartan's First Criterion (characterizing solvability). [2]

   h) State and prove Cartan's Second Criterion (characterizing semisimplicity). [6]
2. Let $V$ be a complex vector space, $L$ a Lie algebra over $\mathbb{C}$, $x$ an element of $L$ and $I$ an ideal of $L$.

a) Define the lower central series of $L$. [1]

b) What is a nilpotent Lie algebra? [1]

c) Prove that if $L \neq \{0\}$ is a nilpotent Lie algebra, then $Z(L) \neq \{0\}$. [2]

d) Give an example of a nilpotent non-abelian Lie algebra. (Explain) [3]

e) If both $I$ and $L/I$ are nilpotent, is $L$ nilpotent? (Explain) [4]

f) If $L$ is a Lie subalgebra of $gl(V)$, explain what it means:
   (i) for $x$ to be nilpotent [1]
   (ii) for $x$ to be ad-nilpotent [1]

g) Suppose that $L$ is a Lie subalgebra of $gl(V)$ and $x \in L$ is nilpotent. Sketch a proof of the fact that $x$ is ad-nilpotent. [4]

h) State Engel’s Theorem for the case when $L$ is a subalgebra of $gl(V)$. [2]

i) State and prove the general Engel’s Theorem (you are allowed to use the previous step). [6]
3. Let $L$ be a Lie algebra over $\mathbb{C}$ and $ad : L \to gl(L)$ the adjoint map defined so that for every $x \in L$, $ad(x) = [x, \cdot]$. 

a) Show that $ad$ is a homomorphism of Lie algebras and find its kernel. [4]

b) Explain what is a derivation $\delta$ of $L$. [2]

c) Show that $Der(L) := \{\delta \mid \delta$ is a derivation of $L\}$ is a Lie subalgebra of $gl(L)$. [4]

d) Show that for all $x \in L$, $ad(x) \in Der(L)$ and $ad(L) \subseteq Der(L)$. [3]

e) Show that for every $x \in L$ and $\delta \in Der(L)$, $[\delta, ad(x)] = ad(\delta(x))$. [2]

f) Show that $ad(L)$ is an ideal of $Der(L)$. [2]

g) Suppose now that $L$ is semisimple and let $M := ad(L)$.

(i) Show that $M$ is semisimple. [2]

(ii) Let $k$ be a Killing form on $Der(L)$ and $M^\perp$ defined with respect to this form. Show that $M \cap M^\perp = \{0\}$ and $Der(L) = M \oplus M^\perp$ (you are allowed to use statements from Problem 1 if necessary). [4]

(iii) Prove that $ad(L) = Der(L)$. [2]
4. Let $E$ be a Euclidean space.

a) Explain what it means for $R \subseteq E$ to be an abstract root system in $E$. [2]

b) Define a base $B$ of an abstract root system $R$ in $E$. [2]

c) What is the Cartan matrix and the Dynkin diagram of a root system? [2]

d) Consider a root system of type $A_2$. Choose a base $B$ for $A_2$. Write all the elements of this root system with respect to $B$, i.e., as linear combinations of elements of $B$. Draw its Dynkin diagram, and write its Cartan matrix. [4]

e) Write Serre’s relations for the root system of type $A_2$. [5]

f) In view of Serre’s Theorem, let us consider a complex Lie algebra $L$ corresponding to $R = A_2$.

(i) Write a Cartan decomposition of $L$. [2]

(ii) What is $\dim_{\mathbb{C}}(L)$? [2]

(iii) Show that $L \cong sl(3, \mathbb{C})$. (Hint: use a standard basis for $sl(3, \mathbb{C})$.) [6]
5. Determine whether the following statements are true or false. If true, give a brief justification; if false, give a counterexample. (You will get zero marks if you do not provide either a justification or a counterexample.)

a) Every Lie subalgebra of solvable Lie algebra is solvable. [3]
b) If $L$ is a proper Lie subalgebra of $sl(2, \mathbb{C})$, then $L$ is solvable. [4]
c) There exists a complex simple Lie algebra $L$ of dimension 5. [4]
d) If $L$ is a semisimple Lie algebra and $\Phi$ is its root system, and $\alpha, \beta \in \Phi$ with $\beta \neq -\alpha$, then $[L_\alpha, L_\beta] \neq \{0\}$ implies $\alpha + \beta \in \Phi$. [4]
e) $L = sl(2, \mathbb{C})$ has two distinct complex irreducible representations of dimension 2010. [2]
f) If $L$ is a Lie algebra that can be written as a direct sum of its simple ideals, then $L$ is semisimple. [4]
g) $gl(2, \mathbb{C})$ is a semisimple Lie algebra. [4]